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Two Interpretations of the Covering Law

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Two completely different physical interpretations of the covering law are presented. One is based on an idea of Jauch and Piron, who tried to interpret this important property of physical theories by using ideal measurements of the first kind. The other uses the notion of degree of incompatibility, which arises naturally if a physically reasonable measure of incompatibility of "yes–no" experiments is assumed to exist.

1. INTRODUCTION

In this paper physical theories—or simply theories—are orthomodular atomic lattices, eventually complete. Any physical discussion in the framework of a given theory uses the standard interpretation of its elements as "yes–no" experiments or as physical quantities having only two possible values. The elements of a theory will be called tests.

In purely mathematical terms a theory is a triple $(L, \leq, \perp)$, where L is a nonempty set, "$\leq$" an order relation, and "$\perp$" an orthocomplementation on L. For any two elements $a, b \in L$, by $a \wedge b$ and $a \vee b$ are denoted the meet and the join of a, b, respectively. If $A \subseteq L$, then $\wedge A$ and $\vee A$ denote the meet, respectively, the join of A. For any $a \in L$ we denote by $a^{\perp}$ its orthocomplement. $\Omega(L)$ denotes the set of atoms of L.

For the smallest and the greatest elements of L we use the notations 0 and 1, respectively. If $R \subseteq L \times L$ is a relation, we will write aRb or $(a, b)R$ instead of $(a, b) \in R$.

Let us now consider L a theory. It is well known that L may be represented as a lattice of projectors in an appropriate Hilbert space only if L satisfies the covering law. This means that $a \in L$, $\alpha \in \Omega(L)$, and $a \wedge \alpha = 0$ imply $a \lessdot \alpha \vee a$ (here $a \lessdot b$ means that b covers a or, equivalently, a is covered by b). The

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geometric interpretation of the covering law is clear, but it comes after L was represented as an orthomodular lattice of orthogonal projectors or, equivalently, closed subspaces cf a Hilbert space. Therefore, it is of an obvious interest to find out if there are physical interpretations of this important property.

In this work we present two physical interpretations of the covering law. The first comes from an idea of Jauch and Piron (1969), which was later put in a more precise mathematical form by Ochs (1972). It uses the so-called ideal measurements of the first kind, which are objects defined in traditional quantum mechanics. Following the empirical definition of ideal measurements of the first kind given by Jauch and Piron, Ochs obtained the mathematical objects describing them in a theory. Then, by using two axioms, he proved that a theory must satisfy the covering law. In their treatment Jauch, Piron, and Ochs used only pure states, that is, those states that may be finally identified with the atoms of the considered theory, so that measurements appear as mappings from the set of pure states into itself. We change the formalism by considering as states the generalized probabilities on theories (see Section 2), so that the ideal measurements of the first kind in Ochs' sense may in principle transform pure states into mixed states. In our approach no special physical axioms are necessary for proving that a physical theory satisfies the covering law. It can be proved that in any theory one may construct ideal measurements of the first kind in Ochs' sense for all tests, some of them also having mixed states among their values. Then, if *all* possible ideal measurements of a theory are supposed—according to the traditional representations—to transform pure states into pure states, we get that in the considered theory the covering law holds. In Section 2 we prove that all theories whose ideal measurements of the first kind have the properties required by traditional quantum mechanics satisfy the covering law.

The second interpretation presented in this paper uses a completely different physical argument. It is based on the natural assumption that there exists a measure of incompatibility of tests. In other words, every physical theory must offer a possibility to compare any two pairs of tests in terms of their empirical incompatibility (the empirical compatibility of tests is defined in Section 2). If this idea is accepted, then it results that on any physical theory we must be able to define a mathematical object called degree of incompatibility. This notion is introduced in Section 3. There it is proved that any theory having a degree of incompatibility defined on it satisfies the covering law. It is also proved that for a large class of theories satisfying the covering law a degree of incompatibility may be constructed in a natural way.

2. MEASUREMENTS AND THE COVERING LAW

Intuitively, any state represents a mode of preparation. In other words, a mode of preparation is a sequence of operations whose result is, in principle, a well-characterized entity called a state. We accept that any test may be measured in any state. A measurement of a test a is an experimental procedure which permits us to decide if a gives the answer "yes" or the answer "no" in any arbitrarily fixed state. Any measurement is considered to be free of subjective errors. It is well known that any measurement of a test in a state changes that state. This fact results easily if we take into account that the state before measurement is a sequence of operations and the measurement itself may be thought of as a new operation, which, "added" to this sequence, gives the state after measurement. This simple observation will be used for obtaining the mathematical definition of a measurement.

A test is said to be true in a given state if the answer "yes" is surely obtained when it is measured in that state. Similarly we may define a false test in any given state. It is very important to realize that a test which is not false (true) in a state is not necessarily true (false) in that state.

We say that a state is pure if it is completely determined by the set of all tests which are true in that state. If this condition is not satisfied, then the state is said to be mixed.

Two tests a and b are said to be empirically compatible if there exists a measurement which measures both a and b in any state.

We are now completely prepared for defining the notion of an ideal measurement of the first kind. First of all it is clear that, since any measurement of a test changes the state, it may be described by a mapping from a given family of states into another family of states. Then, according to the representations of traditional quantum mechanics, an ideal measurement is characterized by the following two properties:

(i) Any ideal measurement transforms pure states into pure states.
(ii) If b and a are compatible tests and b is true in a state s, then any ideal measurement of a transforms s into a state in which b is also true.

A measurement of a is said to be of the first kind if it satisfies the following condition:

(iii) Any state in which a is not false is transformed by the measurement into a state in which a is true.

We give below formal definitions of states and measurements. Ideal measurements of the first kind will be called simply measurements since no other kind of measurement appears in our work.

Let L be a complete theory.

Definition 1. We say that p: $L \rightarrow [0, 1]$ is a state on L if the following two conditions hold:

(1i) The restriction of p to any Boolean orthosublattice of L is a probability.
(1ii) $(a_i)_{i \in I}$, $a_i \in L$, $p(a_i) = 1 \ \forall i$ implies $p(\wedge_i a_i) = 1$.

There are simple examples which prove that the property (1i) does not imply the property (1ii). The set of all states on L will be denoted by S_L. We will suppose also that L has the following property: $\forall \alpha \in \Omega(L)$, there exists a unique state, denoted by δ_α, such that $\delta_\alpha(\alpha) = 1$. The states δ_α will sometimes be called Dirac states.

A set $F \subseteq L$ is called a filter if the following conditions are fulfilled:

(f1) $0 \notin F$.
(f2) $a \in F$, $a \leq b \Rightarrow b \in F$.
(f3) $a_i \in F$, $i \in I \Rightarrow \wedge_i a_i \in F$.

Given a state p, we define the set $A_p = \{a \in L; p(a) = 1\}$. The elements of A_p correspond to the tests which are true in the state p. The set A_p is a filter since it satisfies the conditions (f1)–(f3). It is clear that the state p is completely determined by A_p if and only if $A_p \subseteq A_{p'} \Rightarrow p = p'$. This implication will be considered as the formal definition of a pure state. The next proposition is a characterization of pure states.

Proposition 1. A state p is pure if and only if there exists $\alpha \in \Omega(L)$ such that $p = \delta_\alpha$.

Proof. A filter which is maximal with respect to (f1)–(f3) is called an ultrafilter. Let us observe first that a filter F is an ultrafilter if and only if $\wedge F \in \Omega(L)$. Consequently, if F is an ultrafilter and $\wedge F = \alpha$, then $F = A_{\delta_\alpha}$. Suppose now that p is a pure state and F an ultrafilter with the property $A_p \subseteq F$. Since $F = A_{\delta_\alpha}$ ($\alpha = \bigwedge F$) and p is pure, we obtain $p = \delta_\alpha$. Conversely, let α be an atom. Then δ_α is a pure state. Indeed, let p be a state such that $A_{\delta_\alpha} \subseteq A_p$. Since A_{δ_α} is an ultrafilter, we have $A_{\delta_\alpha} = A_p$, so that $p(\alpha) = 1$. Taking into account that there exists only one state having the value 1 on the atom α, we obtain $p = \delta_\alpha$. It results that δ_α is a pure state and the proposition is completely proved.

The set of all pure states which, according to Proposition 1, is the set of all Dirac states, will be denoted by P_L.

It is a physically justified fact that the empirical compatibility is mathematically described by the relation $C \subseteq L \times L$, which is the commutativity on L and is defined as follows: $(a, b)C \Leftrightarrow a = (a \wedge b) \vee (a \wedge b^\perp)$ (Ivanov, 1991). By using this fact we can define the measurements of L.

Definition 2. We say that the mapping M: $P_L - \{\delta_\alpha; \alpha \leq a^\perp\} \to S_L$ is a measurement of a if the following conditions are satisfied:

(mi) $\delta_\alpha(b) = 1$ and $(b, a)C \Rightarrow (M(\delta_\alpha))(b) = 1$.
(mii) $(M(\delta_\alpha))(a) = 1$.

The conditions (mi) and (mii) represent the exact translation in the language of theories of the empirical conditions (ii) (ideal measurements) and (iii) (measurements of the first kind). Indeed, we saw that $p(a) = 1$ means that a is true in the state p. On the other hand $\delta_\alpha(a) \neq 0$ for all δ_α in the domain of M since if we admit $\delta_\alpha(a) = 0$, then $\delta_\alpha(a^\perp) = 1$, which implies $\delta_\alpha(\alpha \wedge a^\perp) = 1$, so that $\alpha \wedge a^\perp > 0$ and we get $\alpha \leq a^\perp$, absurd.

Given a a test, the set $F_a = \{b \in L;\ b \geq a\}$ is called the filter generated by a ($a \neq 0$).

Proposition 2. M: $P_L - \{\delta_\alpha;\ \alpha \in a^\perp\} \to S_L$ is a measurement for $a \neq 0$ if and only if $F_{(\alpha \vee a^\perp)\wedge a} \subseteq A_{M(\delta_\alpha)}$.

Proof. We will prove first that, if $\alpha \nleq a^\perp$, then $(\alpha \vee a^\perp) \wedge a > 0$. Indeed, $\alpha \vee a^\perp > a^\perp$, otherwise $\alpha \vee a^\perp = a^\perp \Rightarrow \alpha \leq a^\perp$, impossible. Therefore, there exists $q > 0$, $q \leq a$, such that $\alpha \vee a^\perp = q \vee a^\perp$. Now it results easily that $(\alpha \vee a^\perp) \wedge a = q > 0$.

Let M: $P_L - \{\delta_\alpha;\ \alpha \leq a^\perp\} \to S_L$ be a measurement for $a > 0$ and $\alpha \in \Omega(L)$, $\alpha \nleq a^\perp$. Then from the obvious properties $(\alpha \vee a^\perp, a)C$ and $\delta_\alpha(\alpha \vee a^\perp) = 1$ we get $(M(\delta_\alpha))(\alpha \vee a^\perp) = 1$. By using (lii) we obtain $(M(\delta_\alpha))((\alpha \vee a^\perp) \wedge a) = 1$, so that $F_{(\alpha \vee a^\perp)\wedge a} \subseteq A_{M(\delta_\alpha)}$. Conversely, suppose that this inclusion is true for a function M: $P_L - \{\delta_\alpha;\ \alpha \leq a^\perp\} \to S_L$. Then, from $a \geq (\alpha \vee a^\perp) \wedge a$ and $(M(\delta_\alpha))((\alpha \vee a^\perp) \wedge a) = 1$, we get $(M(\delta_\alpha))\ (a) = 1$, which proves (mii). Take now $b \in L$ such that $\delta_\alpha(b) = 1$ and $(a, b)C$. Then $b \geq b \wedge a = (a^\perp \vee b) \wedge a \geq (a^\perp \vee \alpha) \wedge a$, which implies $(M(\delta_\alpha))(b) = 1$ and the property (mi) is proved.

By using this result we can prove the existence of measurements for any element of an arbitrarily given theory.

Proposition 3. Let $(L, \leq, \perp)$ be a theory and $a > 0$. Then the following assertions are true:

(i) There exists a measurement for a.

(ii) If $(\alpha \vee a^\perp) \wedge a \notin \Omega(L)$ for an atom $\alpha \nleq a^\perp$, then there exists a measurement for a having mixed states among its values

Proof. (i) We define a mapping M: $P_L - \{\delta_\alpha;\ \alpha \leq a^\perp\} \to S_L$ as follows: for any $\alpha \nleq a^\perp$ we choose an atom $\beta_\alpha \leq \Omega(L)$, $\beta_\alpha \leq (\alpha \vee a^\perp) \wedge a$ and put $M(\delta_\alpha) = \delta_{\beta_\alpha}$. The mapping M has obviously the property $F_{(\alpha \vee a^\perp)\wedge a} \subseteq A_{M(\delta_\alpha)}$ and it is sufficient to apply Proposition 2.

(ii) If $(\alpha \vee a^\perp) \wedge a \notin \Omega(L)$ for $\alpha \nleq a^\perp$, we find $\beta_1, \beta_2 \in \Omega(L)$, $\beta_1 \perp \beta_2$, $\beta_1, \beta_2 \leq (\alpha \vee a^\perp) \wedge a$. For two positive numbers c_1, c_2 such that $c_1 + c_2 = 1$, we have that $p = c_1\,\delta_{\beta_1} + c_2\,\delta_{\beta_2}$ is a mixed state. Now it is obvious that we can construct a measurement M for a such that $M(\delta_\alpha) = p$.

The central result of this section is a direct conclusion of Proposition 3. We saw already that, according to representations of traditional quantum mechanics, there are no measurements having mixed states as values, so that appropriate theories must consider this fact.

Proposition 4. The theory L satisfies the covering law if and only if all its measurements transform pure states into pure states.

Proof. The proof uses the following result: L satisfies the covering law if and only if for all $a \in L$, $\alpha \in \Omega(L)$, $(\alpha \vee a^{\perp}) \wedge a = 0$ or $(\alpha \vee a^{\perp}) \wedge a \in \Omega(L)$ (Jauch and Piron, 1969).

Suppose that for $a > 0$ there exists a measurement M and an $\alpha \in \Omega(L)$, $M(\delta_{\alpha}) = p$, where p is a mixed state. Then from Proposition 2 we find $\wedge A_p \leq (\alpha \vee a^{\perp}) \wedge a$. But p is a mixed state, so that $\wedge A_p$ is not an atom it results that L does not satisfy the covering law. Suppose now that L does not satisfy the covering law. Then there exists $a > 0$ and $\alpha \in \Omega(L)$ such that $(\alpha \vee a^{\perp}) \wedge a \notin \Omega(L)$. But in this case we may construct, by using Proposition 3, a measurement having mixed states among its values.

The last proposition represents the interpretation of the covering law in "traditional" terms. We mean that, once the characterization of ideal measurements of the first kind as given in the early stage of quantum mechanics is accepted, the covering law as a property of theories is a simple result of purely mathematical manipulations, without any other physical ingredients.

3. COMPATIBILITY AND THE COVERING PROPERTY

In this section we give another interpretation of the covering law, which is entirely based on a deep analysis of the compatibility and incompatibility relations.

Given L a theory and $C \subseteq L \times L$ the compatibility on L, the incompatibility relation is defined naturally by the equality $\bar{C} = L \times L - C$. Obviously, $\bar{C}$ is not empty if and only if L is not Boolean. If L is a Boolean/classical theory (Ivanov, 1992), then L trivially satisfies the covering law.

Let $a, b \in L$ be such that $(a, b)\bar{C}$. In this case, taking account of the definition of C, we may write $a > (a \wedge b) \vee (a \wedge b^{\perp}) = 1(a, b)$ and $b > (b \wedge a) \vee (b \wedge a^{\perp}) = 1(b, a)$. It is easy to prove that $1(a, b)$ is the greatest element smaller than a which is compatible with b. Let us consider the elements $m(a, b) = a - 1(a, b)$ and $m(b, a) = b - 1(b, a)$ (Herţia and Ivanov, 1997). Obviously the element $m(a, b)$ $[m(b, a)]$ is not compatible with $b(a)$. We can say even more: any atom under $m(a, b)$ is not compatible with b. Indeed, $\alpha \in \Omega(L)$, $\alpha \leq m(a, b)$, $(\alpha, b)C$ would imply $\alpha \vee 1(a, b) > 1(a, b)$ and $(\alpha \vee 1(a, b), b)C$, which is impossible because we know that $1(a, b)$ is the greatest element under a which is compatible with b. This fact suggests that *$m(a, b)$ "measures" in a sense the incompatibility of a with b*. Similarly, we can say that *$m(b, a)$ "measures" the incompatibility of b with a*. Therefore, it seems that the elements $m(a, b)$ might be used for defining a degree of incompatibility for all pairs $(a, b) \in L$

$\times$ L. In order to avoid some difficulties which will become clear later, we do not define the degree of incompatibility on the Cartesian product $L \times L$. Instead of $L \times L$ the quotient set $L \times L/J$, where J is an equivalence relation defined below, will be the domain of the degree of incompatibility.

We write $(a, b)J(a', b')$ if and only if $m(a, b) = m(a', b')$. It is obvious that J is an equivalence relation. A very important property of this relation is that it is invariant under automorphisms of L. In order to prove this fact we observe first that the action of an automorphism U: $L \to L$ on $L \times L$ may be defined naturally by the equality $U((a, b)) = (U(a), U(b))$. By using this definition we say that J is invariant under the automorphism U if $(a, b)J(a', b') \Rightarrow (U(a), U(b))J(U(a'), U(b'))$. The proof of this implication is given by the following simple proposition.

Proposition 5. $U(m(a, b)) = m(U(a), U(b))$.

Proof. It is sufficient to verify that $U(1(a, b)) = 1(U(a), U(b))$ or, equivalently, that $U(1(a, b))$ is the greatest element under $U(a)$ which is compatible with $U(b)$. Since U is an automorphism, we can write immediately $U(1(a, b)) = U((a \wedge b) \vee (a \wedge b^{\perp})) = (U(a) \wedge U(b)) \vee (U(a) \wedge U(b)^{\perp}) = 1(U(a), U(b))$ and the proposition is proved.

The invariance of the relation J under automorphisms of L is a straightforward consequence of this simple proposition.

We are now completely prepared for defining the degree of incompatibility [in this definition $(a, b)_J$ denotes the equivalence class of the pair (a, b) with respect to the relation J].

Definition 3. A function D: $L \times L/J \to [0, \infty]$ is said to be a degree of incompatibility on L if it has the following two properties:

(3i) $D((a, b)_J) = D((a', b')_J)$ if and only if there exists an automorphism U such that $U((a, b)_J) = (a', b')_J$.

(3ii) $D((a, b)_J) = D((b, a)_J$ for all pairs (a, b).

The function D is well defined since J is invariant under automorphisms. The property (3i) expresses the simple fact that, if the degree of incompatibility is supposed to be a physically significant object, then it must be invariant under automorphisms. Moreover, if two classes $(a, b)_J$ and $(a', b')_J$ have the same degree of incompatibility, then each of them is obtained by applying to the other an appropriate automorphism. The property (3ii) reflects the obvious physical fact that the pairs (a, b) and (b, a) must be equally incompatible.

We will prove now a very important result for our purpose:

Proposition 6. $U((a, b)_J) = (a', b')_J \Rightarrow U(m(a, b)) = m(a', b')$.

Proof. We saw that $\alpha \leq m(a, b)$, $\alpha \in \Omega(L)$ implies $(\alpha, b)\overline{C}$. Consequently, we may write $m(a, b) \wedge b = m(a, b) \wedge b^{\perp} = 0$ and we get $(m(a, b), b) \in (a,$

$b)_J$, $m(m(a, b), b) = m(a, b)$. On the other hand, we have $(m(a, b), b) \in (a, b)_J \Rightarrow U(m(a, b), b) \in (a', b')_J$. By combining these facts, we get $m(U(m(a, b)), U(b)) = m(m(U(a), U(b)), U(b)) = m(a', b')$.

Corollary. Let L be a theory having the property that there exists a degree of incompatibility defined on it. Then for any pair $(a, b) \in L \times L$ there exists an automorphism U of L such that $U(m(a, b)) = m(b, a)$.

Proof. Let D be a degree of incompatibility on L. Since $D((a, b)_J) = D((b, a)_J)$, we may find an automorphism U such that $U((a, b)_J) = (b, a)_J$ and it remains to use Proposition 6.

Remark. Now it becomes clear why we prefered to use the set $L \times L/J$ instead of $L \times L$. Indeed, when the hypothesis of the Corollary holds, we may always write $U((a, b)_J) = (b, a)_J$, but the equality $U((a, b)) = (b, a)$ is not generally valid.

It is natural to expect that those theories which have degrees of incompatibility defined on them also have certain special geometric properties. We will prove below that this is indeed so: any such theory satisfies the covering law. In order to prove this fact, let us consider the special relation $\overline{C}_1 \subseteq \overline{C}$ defined as follows: $(a, b)\overline{C}_1 \Leftrightarrow m(a, b) \in \Omega(L)$. According to the Corollary, if there exists a degree of incompatibility on L, then we may find an automorphism U such that $U(m(a, b)) = m(b, a)$. Therefore, if $m(a, b)$ is an atom, then $m(b, a)$ is also an atom. It is obvious that $(a, b)\overline{C}_1$ if and only if $(a \wedge b) \vee (a \wedge b^\perp) \lessdot a$. The existence of a degree of incompatibility on L implies obviously the symmetry of the relation $\overline{C}_1$. The following theorem is the central point of this section.

Theorem. The theory L satisfies the covering law if and only if the relation $\overline{C}_1$ is symmetric.

Proof. Suppose that $\overline{C}_1$ is symmetric and consider $\alpha \in \Omega(L)$, $a \in L$, such that $\alpha \wedge a = 0$. If $\alpha \leq a^\perp$, then from the general properties of orthomodular lattices we get $a \lessdot \alpha \vee a$, so that we have to consider the case $\alpha \wedge a^\perp = 0$ only. The set $A_\alpha = \{a \in L;\ a \wedge \alpha = a^\perp \wedge \alpha = 0\}$ has obviously the property $a \in A_\alpha \Rightarrow a^\perp \in A_\alpha$. We will prove now that $a \in A_\alpha \Rightarrow a^\perp \lessdot a^\perp \vee \alpha$. Indeed, from $(\alpha \wedge a) \vee (\alpha \wedge a^\perp) = 0$ it results that $(\alpha, a)\overline{C}_1$ and, since we have also $(a, \alpha)\overline{C}_1$, we get $(a \wedge \alpha) \vee (a \wedge \alpha^\perp) \lessdot a \Rightarrow a \wedge \alpha^\perp \lessdot a \Rightarrow a^\perp \lessdot a^\perp \vee \alpha$. But $a \in A_\alpha \Rightarrow a^\perp \in A_\alpha$, so we get immediately $a \lessdot a \vee \alpha$. Conversely, suppose that L has the covering property and let us prove that $\overline{C}_1$ is symmetric. Let $(a, b) \in L \times L$ be a pair such that $(a, b)\overline{C}_1$ and the atom $\alpha = a - 1(a, b)$. We will prove first the equality

$$b \vee \alpha = (a \vee b) \wedge (a^\perp \vee b) \tag{*}$$

Since the atom α and the element $1(a, b)$ are orthogonal, we may write $\alpha \leq$

$(a^{\perp} \vee b^{\perp}) \wedge (a^{\perp} \vee b)$, so that $\alpha \leq a^{\perp} \vee b$. Then, since $\alpha \leq a$, we get $\alpha \leq a \vee b$ and, by combining these facts, we obtain the inequality $b \vee \alpha \leq (a \vee b) \wedge (a^{\perp} \vee b)$. In order to prove (*) it is sufficient to verify the equality $(a \vee b) \wedge (a^{\perp} \vee b) \wedge (b \vee \alpha)^{\perp} = 0$. Taking into account the relation $a = \alpha \vee 1(a, b)$, we obtain

$$
\begin{aligned}
&(a \vee b) \wedge (b \vee \alpha)^{\perp} \\
&\quad = [b \vee (a \wedge b) \vee (a \wedge b^{\perp}) \vee \alpha] \wedge (b \vee \alpha)^{\perp} \\
&\quad = [(b \vee \alpha) \vee (a \wedge b^{\perp})] \wedge (b \vee \alpha)^{\perp} \\
&\quad = (a \wedge b^{\perp}) \wedge (b \vee \alpha)^{\perp} = a \wedge b^{\perp}
\end{aligned}
$$

and (*) is proved. It remains to verify that $(b, a)C_1$ or, equivalently, $(a \wedge b) \vee (a^{\perp} \wedge b) \lessdot b$. We observe first that $\alpha \wedge b = 0$ since $\alpha = m(a, b)$ and we saw that all atoms under $m(a, b)$ are not compatible with b. Since L satisfies the covering law, the relation (*) gives $b \lessdot (a \vee b) \wedge (a^{\perp} \vee b)$ or, equivalently, $(b^{\perp}, a)C_1$. Since C_1 has also the property $(a, b)C_1 \Rightarrow (a, b^{\perp})C_1$, we may write $(a, b)C_1 \Rightarrow (a, b^{\perp})C_1 \Rightarrow ((b^{\perp})^{\perp}, a)C_1 \Rightarrow (b, a)C_1$ and the theorem is completely proved.

It has been seen that the definition of the degree of incompatibility has a quite transparent physical basis. Taking account of this fact, we may affirm that the theorem we just proved represents a physical interpretation of the covering law.

Although this interpretation of the covering law looks quite natural, it will be complete after it will be proved that for a theory which satisfies the covering law a degree of incompatibility can be constructed. Our discussion will be restricted to those pairs $(a, b) \in L \times L$ having the property that $m(a, b)$ is a finite element (an element of L is finite if it has an finite orthogonal decomposition in atoms). We will construct a degree of incompatibility for a theory L which has some special properties. In order to formulate them we need the notion of basis of the theory L. Any orthogonal decomposition in atoms of the element $1 \in L$ is called a basis of L. It can be proved that any orthogonal family of atoms of L is contained in a basis.

Any one of the theories considered below has the following property: given L such a theory, B, B' two of its bases, and f: $B \to B'$ a bijective mapping, there exists an automorphism U of L such that $U(b) = f(b)$ for all $b \in B$. This assumption is valid if L is the lattice of all orthogonal projectors in a Hilbert space. The well-known Piron representation theorem for complete theories assures the validity of the above enounced property for almost all interesting theories. In order to avoid physically irrelevant mathematical problems we will assume also that the theories considered below have countable bases.

Let L be a theory satisfying the aforementioned properties and the covering law. We will show that it is possible to construct in a quite natural way an incompatibility degree on L.

It has been seen that any equivalence class $(a, b)_J$ is characterized by the element $m(a, b)$ in the sense that $m(a, b)$ is directly related to a possible measure of incompatibility for the pairs (a', b') belonging to $(a, b)_J$. Let us denote by D the degree of incompatibility we intend to construct. Since for any compatible pair (a, b) we have $m(a, b) = 0$, it results that all compatible pairs are J-equivalent. The J-class of all compatible pairs is obviously the relation C, so that it is natural to put $D(C) = 0$. For the pairs (a, b) with the property $m(a, b) \in \Omega(L)$ we put $D((a, b)_J) = 1$. It is intuitively clear that, given a degree of incompatibility D, the inequality $D((a, b)_J) < D((a', b')_J)$ means that the pair (a, b) is "less incompatible" than the pair (a', b'). Since $m(a, b)$ was accepted as a measure of incompatibility of the pair (a, b), it results that there does not exist a pair of tests less incompatible than the pairs belonging to $\overline{C}_1$. Therefore D takes no values in the open interval (0, 1). Taking into account all these facts, it appears as natural to define the function D by the equality $D((a, b)_J) =$ dim $m(a,b)$, where dim a denotes the number of elements of an orthogonal decomposition in atoms of the element a. If $m(a, b)$ is not a finite element, then we write $D((a, b)_J) = \infty$. It is easy to prove that D is a degree of incompatibility. Let us consider (a, b), (a', b') two pairs of tests such that $D((a, b)_J) = D((a', b')_J)$ and $B(a, b)$, $B(a', b')$ orthogonal decompositions in atoms of the elements $m(a, b)$, $m(a', b')$, respectively. We may find two bases B, B' in L such that $B(a, b) \subseteq B$, $B(a', b') \subseteq B'$. It is easy to construct a bijective function f: $B \rightarrow B'$ with the property $f(B(a, b)) = B(a', b')$. Since there exists an automorphism U with the properties $U(B) = B'$ and $U(b) = f(b)$ for all $b \in B$, we get easily $U(m(a, b)) = m(a', b')$. Therefore D is a degree of incompatibility if we can prove that $D((a, b)_J) = D((b, a)_J)$. But this equality results from the covering law (Herţia and Ivanov, 1997).

4. CONCLUSION

In this work we obtained a serious motivation for considering the covering law to be a statement with a clear physical origin. Indeed, both interpretations given here use notions considered fundamental in early quantum mechanics. The first states that a theory must be able to describe ideal measurements of the first kind as mappings transforming pure states into pure states. The second is based on the assumption that a theory must offer the possibility to compare the incompatibilities of any two pairs of tests. Such a possibility appears after a careful analysis of the incompatibility relation, which leads naturally to the notion of degree of incompatibility. Finally, the fact that the covering law may be interpreted by following two independent physical approaches confirms once again the physical roots of this important property.

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